Development of Numerical Methods for Geodynamo and Mantle Convection Simulations

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By combining the ideas of pseudo-compressibility and local time stepping together with multigrid method, we have developed a new numerical algorithm for mantle convection simulation with strongly variable viscosity, which is a characteristic feature of mantle material, and implemented the algorithm to a mantle convection simulation code in box geometry. We have also devised a new grid system for fluid simulations in spherical shell geometry. This new spherical grid, named Yin-Yang grid, is based on the overset (or Chimera) grid methodology. The Yin-Yang grid is applied to geodynamo and mantle convection simulations in spherical shells with good success.

Keywords: mantle convection, geodynamo, multigrid method, finite difference method, spherical shell, Yin-Yang grid

1. Introduction

We have focused this year on research and development of new numerical algorithms and codes that enable us to perform large scale computer simulations of mantle convection and geodynamo by Earth Simulator.

2. Development of New Spherical Grid: Yin-Yang Grid

Since the finite difference method enables us to make highly optimized programs for massively parallel computers, we exploit the possibility of the finite difference method for simulations in spherical shell geometry with radius r ($r_i \leq r \leq r_o$), colatitude θ ($0 \leq \theta \leq \pi$), and longitude ϕ ($0 \leq \phi < 2\pi$).

Because there is no grid mesh that is orthogonal all over the spherical surface and, at the same time, free of coordinate singularity or grid convergence, we decompose the spherical surface into subregions. The decomposition, or dissection, enables us to cover each subregion by a grid system that is individually orthogonal and singularity-free. This *divide-and-rule* approach has been used with good success in the computational aerodynamics that incorporates complex geometry of aircraft's body with wings/stores/blades.

The dissection of the computational domain generates internal border or boundary between the subregions. There are two different approaches to handle the internal boundaries. One is the patched grid method[10] and the other is the overset grid method[3]. In the patched grid approach, the subdomains contact one another without any overlap on their borders. In the overset grid method, on the other hand, the subdomains partially overlap one another on their borders. The overset grid is also called as overlaid grid, or composite overlapping grid, or Chimera grid[13]. The validity and importance of the overset approach in the aerodynamical calculations was pointed out by Steger[12]. Since then this method is widely used in this field. It is now one of the most important grid techniques in the computational aerodynamics.

We have proposed a new overset grid for spherical geometry that is named "Yin-Yang grid"[6] after the symbol for yin and yang of Chinese philosophy of complementarity. The Yin-Yang grid is composed of two identical component grids. Compared with other possible spherical overset grids, the Yin-Yang grid is simple in its geometry as well as metric tensors. A remarkable feature of this overset grid is that the two identical component grids are combined in a complemental way with a special symmetry.

The Yin-Yang grid in its most basic shape is shown in Fig. 1. It has two component grids that are geometrically identical (exactly the same shape and size); see Fig. 1(a). We call the two component grids "Yin grid" (or *n*-grid) and "Yang grid" (or *e*-grid). They are combined to cover a spherical shell with partial overlap on their borders as shown in Fig. 1(b). Each component grid is in fact a part of the latitude-longitude grid: A component grid, say Yin grid, is defined in the spherical polar coordinates by its low latitude region of 90˚ (45˚ N and S) about the equator and 270˚ in the azimuthal direction. Another component grid, Yang grid, is defined by the same rule but in different spherical coordi-

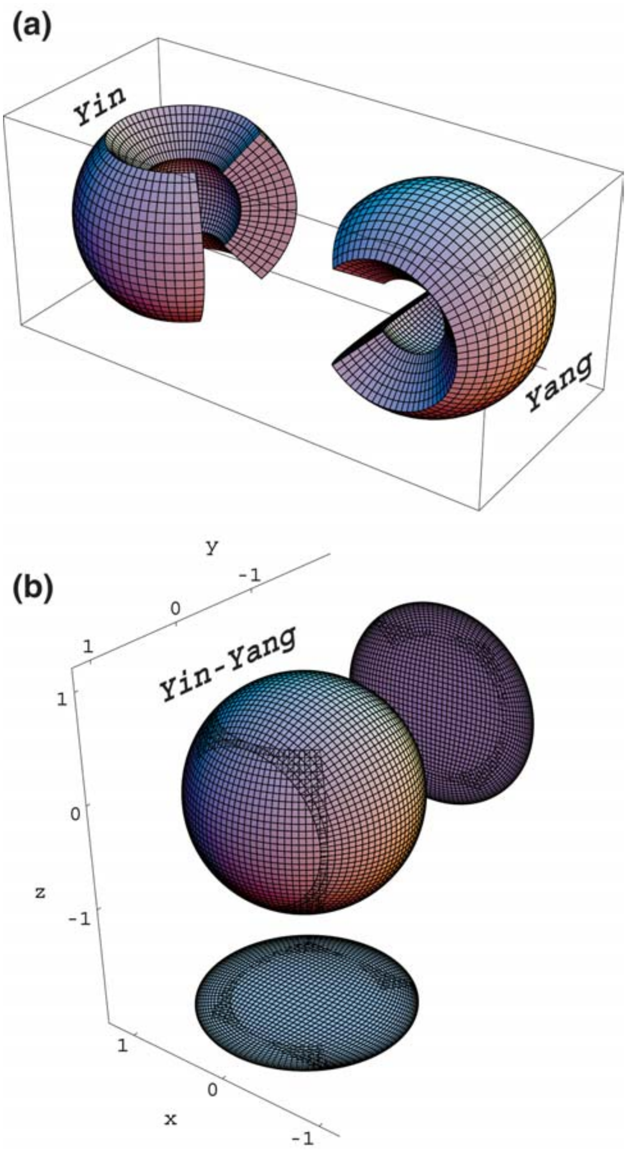

Fig. 1 A spherical overset grid named "Yin-Yang grid".

nates that is perpendicular to the original one; the axis of the Yang grid's coordinates is located in a equator of the Yin grid's coordinates. The relation between Yin coordinates and Yang coordinates is denoted in the Cartesian coordinates by

$$(x^e, y^e, z^e) = (-x^n, z^n, y^n) , \qquad (1)$$

where (x^n,y^n,z^n) are Yin's Cartesian coordinates and (x^e,y^e,z^e) are Yang's. Note that the above transformation is complemental:

$$(x^n, y^n, z^n) = (-x^e, z^e, y^e). \qquad (2)$$

Since the two component grids are identical and their geometrical positions are complemental, we can make concise programs: Suppose a grid point (θ_j^n , ϕ_k^n) on Yin grid's horizontal border at index position (j, k) (e.g., $j = 1$). Its value should be determined by an interpolation from its neighbor points, or stencils, of Yang grid with interpolation coefficients that are determined by relative position of (θ_j^n , ϕ_k^n) in the stencils. Note that exactly the same interpolation coefficients and relative stencils are used to set the value of corresponding grid point (θ_j^e , ϕ_k^e) at (j, k) of Yang's border, since the geometrical relations between Yin grid and Yang grid are symmetric. In other words, we can make use of one interpolation routine for two times (for Yin grid and for Yang grid) to set the horizontal boundary conditions. Note also that the metric tensors at a bulk grid point at (j, k) of Yin grid is a function of its position (θ_j^n , ϕ_k^n) in Yin's coordinates, and the metric tensors at corresponding point (θ_j^e , ϕ_k^e) in Yang grid are exactly the same. Therefore, we can call one subroutine of fluid solver for two times for Yin grid and Yang grid.

Another advantage of the Yin-Yang grid resides in the fact that the component grid is nothing but a (part of) latitude-longitude grid. We can directly deal with the equations to be solved with vector form in the usual spherical polar coordinates; $\{v_r, v_\theta, v_\phi\}$. The analytical form of metric tensors in the spherical coordinates are familiar. We can directly code the basic equations in the program as they are formulated in the spherical coordinates. We can make use of various resources of mathematical formulas, program libraries, and tools that have been developed in the spherical polar coordinates.

3. Dynamo Simulation by Yin-Yang Grid

We have developed a geodynamo simulation code using a finite difference method in the Yin-Yang grid. We have confirmed that the magnetic field generation by the magnetohydrodynamic dynamo process in the core is successfully reproduced by the Yin-Yang grid method. Fig. 2 shows the three-dimensional convection flow.

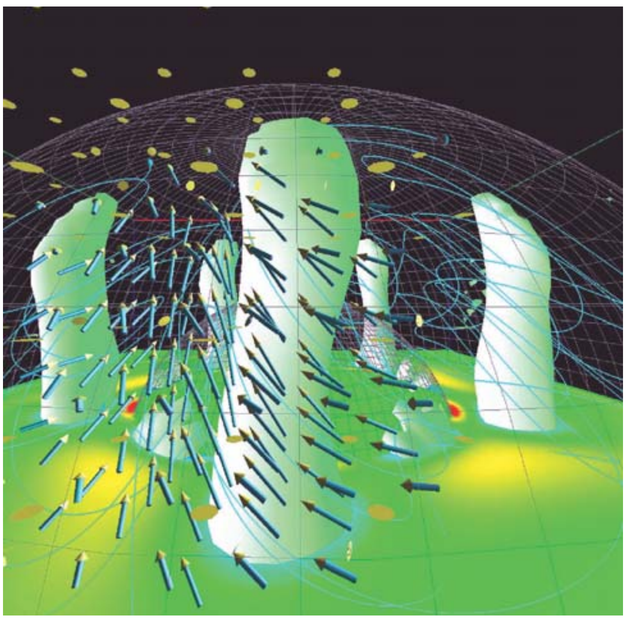

Fig. 2 Core convection simulation by Yin-Yang grid.

4. Mantle Convection Simulation by Yin-Yang Grid

We have developed a new numerical simulation code to solve the thermal convection of a Boussinesq fluid with infinite Prandtl number using Yin-Yang grid[16]. The non-dimensional equations of mass, momentum and energy are solved by the finite difference discretization with second-order accuracy. Using the Yin-Yang grid method, we simultaneously solve these equations for each component grid. We use the collocated grid method; all the primitive variables, velocity, pressure, and temperature, are defined on the same grid points. The SIMPLER method[9, 5] is applied to solve velocity and pressure. The Crank-Nicolson method is used in the energy equation for the time stepping. The upwind difference method is applied for the advection term in the energy equation. We use a Successive Over-Relaxation (SOR) method as the iterative solver required in the SIMPLER method. The horizontal boundary values of each component grid are determined by linear interpolation of the another component grid. The interpolation is taken at each SOR iteration. We performed benchmark tests with published numerical codes and confirmed the validity of our code.

We have focused on the convection at Rayleigh numbers up to 10^6 to confirm the applicability of the Yin-Yang grid. When $Ra_{bot} = 10^5$, the convection patterns become weakly time-dependent; the geometrical symmetry in this Rayleigh number is broken. This disagrees with the result of Ratcliff et al.[11]. In the isoviscous case with "cubic symmetry", all the six upwelling plumes have the same diameters in our results (Fig. 3). The corresponding case by Ratcliff et al., in which a finite volume scheme on the latitude-longitude grid is used, shows a symmetric pattern about equator and appears to remain in a steady state. These observations suggest that the low Rayleigh number convections around $Ra_{bot} = 10^5$ are numerically affected by coordinate singularity and grid convergence in the latitude-longitude grid method. The large-scale (low degree) convective structures are numerically affected by the poles and the grid convergence. On the other hand, the pole problems are removed in our code by making use of Yin-Yang grid.

The Yin-Yang grid is suitable to solve the mantle convection problems in the spherical geometry because it automatically avoids the pole problems that are inevitable in the latitude-longitude grid. Based on this grid, our code is powerful and unique finite difference based code that can solve both the uniform and the strongly variable viscosity convections.

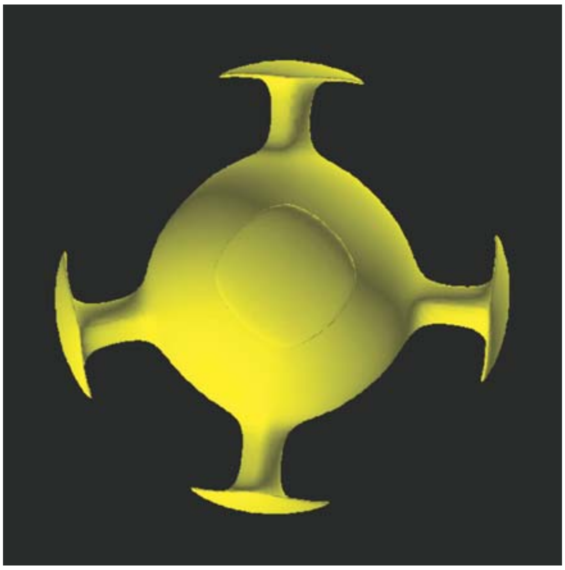

Fig. 3 Mantle convection simulation by Yin-Yang grid.

5. Development of New Algorithm for the Mantle Convection

The major difficulty in numerical simulations of the mantle convection lies in solving the flow field. Since the mantle material is highly viscous[14] ($O(10^{21})$ Pa s), the flow in the mantle is described by a steady-state flow balancing between the buoyancy force, pressure gradient and viscous resistance. In addition, the viscosity of mantle material varies by several orders of magnitude depending on temperature, pressure, and stress[15, 8]. Because of its extreme rheological properties, one must solve ill-conditioned simultaneous equations for velocity and pressure at every timestep. It is very important to develop efficient numerical techniques that can deal with the steady-state flow of highly viscous fluids with a strongly variable viscosity, in order to conduct large-scale simulations of mantle convection.

In this study, we developed a numerical algorithm for solving mantle convection problems with strongly variable viscosity. Equations for conservation of mass and momentum for highly viscous and incompressible fluids are solved iteratively by a multigrid method[1, 2] in combination with pseudo-compressibility[4] and local time stepping techniques. The detail of the algorithm can be found in Kameyama et al.[7].

We implemented this algorithm into a mantle convection simulation program in a three-dimensional rectangular domain, and performed several calculations for mantle convection with strongly variable viscosity. Figure 4 shows a snapshot of thermal convection of mantle with a viscosity variation of 10^5. Benchmark comparisons with previous two-

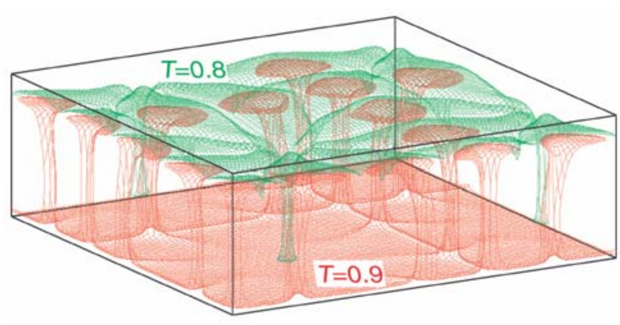

Fig. 4 Simulation of thermal convection of mantle with a strongly variable viscosity in a rectanglar domain.

and three-dimensional calculations revealed that accurate results are obtained even for the cases with viscosity variations of several orders of magnitude. We found that the present method successfully solved the cases with the global viscosity variations up to 10^{10} by adapting the costs of pre- and post-smoothing calculations during the multigrid operations. This program is easily vectorized and parallelized, and we obtained the vector operation ratio of 97.94% and parallelization ratio of 99.53% for the calculations with $720 \times 720 \times 320$ mesh using 15 nodes.

The present method is proved to be suitable for the large-scale simulation of mantle convection compared to the SIMPLER method[9] by carrying out three-dimensional simulations using these two methods; (i) the memory size required for the present method is only a half of that for the SIMPLER method; (ii) the computational time linearly increases with the present method as the number of unknown variables N increases, whereas it increases in proportion to $N^{4/3}$ for the SIMPLER method (when the conjugate gradient method is used for solving the Poisson equations); and (iii) the cases with variable viscosity can be solved with much less computational costs for the present method than for the SIMPLER method.

6. Summary

We have devised a new grid in spherical shell geometry based on the overset methodology. The new overset grid, named Yin-Yang grid, is applied to spherical shell mantle convection and geodynamo simulations with good success. We have also developed a new algorithm for mantle convection simulation by combining pseudo-compressibility and local time stepping method together with the multigrid method. This algorithm is implemented to mantle simulation codes in box geometry.

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